ESTIMATING THE DEGREE OF ANISOTROPY OF ELASTIC MEDIA

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The refraction curves, wave-front geometry, and changes taking place in these characteristics on varying the elastic constants of anisotropic media over wide ranges are analyzed. A quantitative criterion is derived for estimating the number and disposition of the lacunas, the properties of the roots of the characteristic equation, and other important characteristics of the medium.

1. We shall consider media characterized by the following equations of motion under conditions of plane deformation:

$$\begin{aligned} c_1\frac{\partial^2 u}{\partial x^2}+c_2\frac{\partial^2 w}{\partial x\partial z}+c_3\frac{\partial^2 u}{\partial z^2}-\rho\frac{\partial^2 u}{\partial t^2}&=-\rho a_1 f;\\ c_3\frac{\partial^2 w}{\partial x^2}+c_2\frac{\partial^2 u}{\partial x\partial z}+c_4\frac{\partial^2 w}{\partial z^2}-\rho\frac{\partial^2 w}{\partial t^2}&=-\rho a_2 f, \end{aligned} \tag{1.1}$$

where u, w are the displacement vector components along the x and z axes, respectively; c_1, c_2, c_3, c_4 are certain coefficients expressed in terms of the elastic constants of the medium, ρ is the density, t is the time, a_1, a_2 are constants, and f is a certain function of x, z, t.

The equations of motion of elastic anisotropic media reduce to the system (1.1) in four cases.

1. Cubic crystals (three elastic constants). The quantities c_1, c_2, c_3, c_4 are expressed in terms of the elastic constants of the medium in the following way:

$$c_1=c_4=a_{11},\ c_2=a_{12}+a_{44},\ c_3=a_{44}.$$

2. Transversally isotropic media, hexagonal crystals, and certain rhombohedral forms (five elastic constants).

$$c_1=a_{11},\quad c_2=a_{13}+a_{44},\quad c_3=a_{44},\quad c_4=a_{33}.$$

3. Certain forms of tetragonal crystals (six elastic constants),

$$c_1=a_{11},\ c_2=a_{13}+a_{44},\ c_3=a_{44},\ c_4=a_{33}.$$

4. Rhombohedral cyrstals and orthotropic materials (nine elastic constants),

$$c_1=a_{11},\ c_2=a_{13}+a_{55},\ c_3=a_{55},\ c_4=a_{33}.$$

These results follow from those given by Love [1]. The directions of the coordinate axes are also chosen in accordance with [1].

It is assumed that the coefficients of the equations satisfy the conditions of hyperbolic configuration, which in terms of the quantities α, β, γ assume the form [2].

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$$-2\sqrt{\alpha\beta} < \gamma < 1+\alpha\beta,$$

$$\alpha = c_3/c_1, \quad \beta = c_3/c_4, \quad \gamma = 1 + \alpha\beta - \frac{c_2^2}{c_1c_4}. \tag{1.2}$$

The conditions for the positive-definite state of the elastic energy in the case of plane deformation may be written

$$c_1c_4 > (c_2 - c_3)^2 \tag{1.3}$$

and coincide with the conditions required to ensure real roots of the Rayleigh equation.

For triaxial deformation the positive-definite condition contains the complete set of elastic constants and has its own particular form for each of the foregoing classes of elastic symmetry.

The transition from isotropic media (a very special case of elastic media) to those of more general form poses the question as to the nature of the singularities characterizing transient wave fields in the anisotropic material and the possibility of replacing the latter by a certain equivalent isotropic medium, since the solutions of all practical problems take a simpler form in the latter case. For this purpose it is desirable to possess simple quantitative criteria giving a complete representation of the wave picture without having to solve any specific problem. From this point of view problems involving concentrated, pulsed action acquire a special significance, since the closed solution obtained for these systems enable us the study of the foregoing problems very widely [3, 4, 5].

Of all the media satisfying conditions (1.2) and (1.3) we separate out those satisfying the following inequalities [3, 4]:

$$\gamma^2 \geqslant 4\alpha\beta, \ \gamma > \alpha(\beta+1), \tag{1.4}$$
$$[2\beta(1+\alpha) - \gamma(1+\beta)] \geqslant -|\beta - 1|\sqrt{\gamma^2 - 4\alpha\beta};$$
$$\gamma^2 \geqslant 4\alpha\beta, \ \gamma > \beta(\alpha+1), \tag{1.5}$$
$$[2\alpha(1+\beta) - \gamma(1+\alpha)] \geqslant -|\alpha - 1|\sqrt{\gamma^2 - 4\alpha\beta};$$
$$0 < \alpha < 1, \ 0 < \beta < 1. \tag{1.6}$$

We classify these media as the "first group." Subject to conditions (1.4)-(1-6), the roots of the characteristic equation $\mu_n(\theta)$ and $\theta_n(\mu)$ for real values of the argument take only real or purely imaginary values [3, 4].

Investigations show that even for plane deformation the permissible range of values of c_1, c_2, c_3, c_4 is determined, not only by conditions (1.2) and (1.3), but also by the conditions required to ensure the positive-definite nature of the elastic energy in cases of triaxial deformation. Hence, each type of elastic symmetry should be considered separately, despite the fact that under conditions of plane deformation all are described by the same system of equations of motion (1.1).

Let us consider a medium with $c_1 = c_4$ or $\alpha = \beta$, a particular case of this including crystals of the cubic system. If the medium is isotropic, then $\gamma = 2\alpha$. Of the five types of wave-front geometry feasible, the two illustrated in Fig. 1 are realized for plane deformation in the case of media satisfying conditions (1.4) and (1.5). The first is characterized by smooth fronts (Fig. 1a, rock salt). The wave fronts (Fig. 1b, c, d) characterized by four lacunas not lying on the coordinate axes (ice, beryl, sandstone) belong to the second type. Here we are considering the wave fronts arising in an infinite medium from a concentrated pulse source [3].

For $0 < \alpha < 0.75$, an anisotropic medium of the type under consideration ($c_1 = c_4$) may be represented by a single isotropic medium, propagating waves at the same velocities as the former along the x and z axes, i.e., having the same values of the coefficients c_1, c_3, ρ. For such an isotropic medium $c_2 = c_1 - c_3 = c_{\alpha is}$.

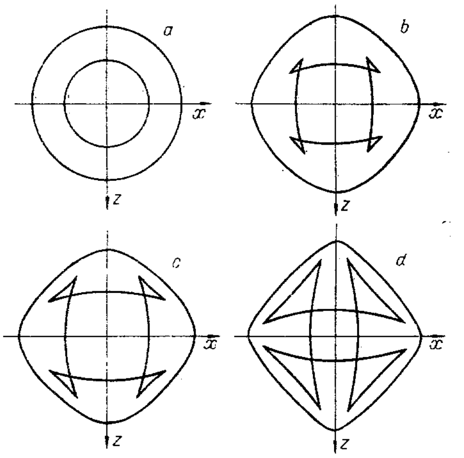

Fig. 1

On the other hand, corresponding to every isotropic medium we have an infinite number (series) of anisotropic media having the same coeffients c_1, c_3, ρ as the latter and only differing in respect to the coefficient c_2. It is thus natural to consider the value of the coefficient

$$\Delta_A = c_{2\,\mathrm{is}}/c_2 = (c_1 - c_3)/c_2$$

as a criterion for comparing the properties of anisotropic media belonging to the series in question with each other and with the isotropic medium generating the series. Allowing for the condition $\gamma^2 \geqslant 4\alpha\beta$, which in the case $\alpha = \beta$ takes the form $\gamma \geqslant 2\alpha$, we find that for media of the first group $\Delta_A \geqslant 1$.

By comparing the configurations of the wave fronts in Fig. 1, we see that for $\Delta_A = 1.107$; $\Delta_{A*} = 1.325$ (rock salt) the wave fronts are smooth, while for $\Delta_A = 1.442$; $\Delta_{A*} = 1.187$ (ice) lacunas appear, increasing in size with increasing Δ_A : $\Delta_A = 2.84$; $\Delta_{A*} = 1.1$ (beryl) and $\Delta_A = 3.85$; $\Delta_{A*} = 1.05$ (sandstone). It is natural to assume that as Δ_A falls from $\Delta_A = 1.442$ the size of the lacunas will diminish; on passing through a certain value $\Delta_A = \Delta_{A*}$ they will vanish, and the fronts will become smooth, as in the case of rock salt. An example of a medium in which $\Delta_A \approx \Delta_{A*}$, is sylvine.

The geometry of the wave fronts obtained for a concentrated pulse source is represented by the envelope of a family of plane waves defined by the equation

$$t - \theta x - \mu_n(\theta) z = 0,$$

where $\mu_n(\theta)$ are the roots of the characteristic equation of the system.

The function $\mu_1(\theta)$ in the range $|\theta| \leqslant a$ and $\mu_2(\theta)$ in the range $|\theta| \leqslant b$, $(a = \sqrt{\rho/c_1},\ b = \sqrt{\rho/c_3})$ define the two corresponding branches of the refraction curve on the μ, θ, plane, which are closely related as regards shape to the geometry of the wave fronts. If the refraction curves are convex, the wave fronts are smooth. The geometry of the wave fronts containing the lacunas represents the case in which the branch of the refraction curve corresponding to $\mu_2(\theta)$ has a point of inflection. Since the refraction curves are of the fourth order, the maximum number of points of inflection is eight, i.e., two in each of the quadrants on the μ, θ plane. Each pair of inflections corresponds to one lacuna, so that in this case four lacunas exist. If conditions (1.4) and (1.5) are satisfied, the wave fronts have the form of Fig. 1b,c,d. (The situation in which the refraction curve corresponding to $\mu_2(\theta)$ has four points of inflection is only possible for media in which $c_1 \neq c_4$). There are no other possibilities for media with $c_1 = c_4$, apart from the case in which points of inflection are completely absent. The instant of passing from smooth wave fronts and convex refraction curves to wave fronts with four lacunas, arranged as Fig. 1b, c, d, corresponds to the moment at which inflections appear on the refraction curve corresponding to $\mu_2(\theta)$ (outer curve). For a specific value this corresponds to a certain critical $\gamma = \gamma_*$. The condition for the development of points of inflection on the $\mu_2(\theta)$ curve gives the following expression for γ_*

$$2\gamma_* = -3(1-\alpha)^2 + (1+\alpha)\sqrt{9\alpha^2 - 14\alpha + 9}. \tag{1.7}$$

Expressing Δ_A, in terms of α and γ we obtain

$$\Delta_A = (1-\alpha)/\sqrt{1+\alpha^2-\gamma},\ \gamma < 1+\alpha^2. \tag{1.8}$$

Substituting the expression for γ_* into (1.8), we obtain a formula for the critical value of $\Delta_{A*}(\alpha)$, corresponding to a transition from wave fronts of the first type ($\Delta_A < \Delta_{A*}$, region II in Fig. 2) to wave fronts of the second type ($\Delta_A > \Delta_{A*}$, region I in Fig. 2). The relationship $\Delta_A = \Delta_{A*}(\alpha)$ defines curve 1 in Fig. 2. The further Δ_A stands in region I from the value Δ_{A*}, corresponding to the specified α, the larger is the region occupied by the lacunas (sandstone).

If we return to the curves of displacements for the problem of a concentrated source in an unlimited medium [3] or for the Lamb problem [4] we see that the same tendency appears. In the case of rock salt the curves of displacements differ little from the corresponding curves for the model isotropic medium chosen in the manner just indicated. For sandstone the differences are very great [3, 4]. If the values of Δ_A lie in the range $1 < \Delta_A \ll \Delta_{A*}$, the anisotropic medium may be replaced by the isotropic medium already indicated. Complete coincidence will occur for $\Delta_A = 1$ (curve 2). Thus in the case of media of the first group Δ_A may serve as a criterion for estimating the size of the region occupied by the lacunas and also the difference between the displacement curves which may be expected to arise on replacing the anisotropic by the isotropic medium when considering plane (and, in the case of transversally isotropic media, axisymmetrical) problems. The results also extend to media not related to cubic crystals, but in which c_1 and c_4 (even if not equal) are of similar values ($\alpha \approx \beta$).

The results obtained related to the case $\Delta_A \geqslant 1$.

2. Let us consider media for which $\Delta_A < 1$ and the condition $\alpha = \beta$ is observed. Subject to the condition $\alpha < 1$ the whole of the permissible range of γ values comprises three intervals:

$$-2\alpha < \gamma < \alpha(\alpha+1), \ \alpha(\alpha+1) < \gamma < 2\alpha; \ 2\alpha \leqslant \gamma < 1+\alpha^2. \tag{2.1}$$

The boundary values of -2α, $1+\alpha^2$ arise from the conditions governing the hyperbolic state. The range $2\alpha \leqslant \gamma < 1+\alpha^2$ is associated with the first-group media already considered. It remains to consider the two first intervals of (2.1). On passing from the values $\gamma \geqslant 2\alpha$ to the values $\gamma < 2\alpha$ of Eqs. (1.4) and (1.5) only one inequality remains in force: $\gamma > \alpha(\alpha+1)$.

The coefficient Δ_A becomes smaller than unity, i.e., there is a transition from region II to region III (through the straight line $\Delta_A = 1$) on the Δ_A, α plane (Fig. 2).

For values of Δ_A in region III the wave fronts are smooth and the refraction curves convex. However, the roots of the characteristic equation $\mu_n(\theta)$ take a different form on the real axis θ and belong to those of the "second type," according to the classification of [5]. The lower boundary of region III is curve 3

$$\Delta^0_{A*} = \sqrt{1-\alpha}.$$

This curve corresponds to values of Δ_A for which inflection points ($\gamma = \alpha(\alpha+1)$) arise on the refraction curve, so that in region IV on the Δ_A, α plane the refraction curves exhibit regions of concavity, as in region I, but in contrast to region I these are situated on the coordinate axes, and hence the wave fronts are characterized by the presence of lacunas intersected by the x and z axes. The roots of the characteristic equation are of the third type according to [5].

The lower boundary of region IV in Fig. 2 (curve 4) is defined by the values of Δ_A associated with $\gamma = -2\alpha$, which are also the minimum values, since for smaller values of Δ_A the hyperbolic conditions are not satisfied. Representatives of media for which the values of Δ_A lie in region IV include copper ($\Delta_A = 0.478$) and the majority of pure metals with cubic lattices, so that these media are quite common in addition to those having Δ_A values in the regions I and II.

For the majority of metals with cubic and hexagonal close-packed lattices, as well as many minerals, the values of α lie in the range $0 < \alpha < 0.5$, as in the cases which we have just been considering. It should be noted that, from the point of view of thermodynamics, isotropic media admit α values in the range $0 < \alpha < 0.75$. Acually, we always have $0 < \alpha < 0.5$, so that the interval $0.5 < \alpha < 0.75$ corresponds, in the case of isotropic solids, to negative values of the Poisson coefficient. Such materials should experience an increase in transverse dimensions on being subjected to tensile strain. No such materials are in fact known [6]. In certain materials with cubic lattice (iron, germanium, potassium, sodium, lithium) and also in beryllium (hexagonal close-packed structure) the value of α lies in the range $0.5 < \alpha < 0.75$.

In the case of a source in an unlimited medium (and the Lamb problem) for media with Δ_A values in the ranges III and IV, the curves of displacements contain a series of sin-

TABLE 1

Element	α	β	Type of lattice	Δ_A	No. of region
Ag	0,37	0,37	C	0,558	IV
Al	0,264	0,264	C	0,884	III
Au	0,216	0,216	C	0,724	IV
Nb	0,117	0,117	C	1,33	II
Ir	0,45	0,45	C	0,623	IV
Mo	0,276	0,276	C	1,08	II
Ni	0,492	0,492	C	0,465	IV
Pb	0,301	0,301	C	0,604	IV
V	0,187	0,187	C	1,147	II
Mg	0,275	0,266	HD	1,136	II
Y	0,312	0,316	HD	1,183	II

TABLE 2

Element	α	β	γ	Δ_A	No. of region
Fe	0,505	0,505	0,075	0,455	IV
K	0,508	0,508	—0,582	0,362	IV
Ge	0,519	0,519	0,468	0,52	IV
C	0,535	0,535	0,862	0,713	III
Na	0,575	0,575	—0,736	0,34	IV
Th	0,635	0,635	—0,247	0,284	IV
Li	0,652	0,652	—1,525	0,212	IV

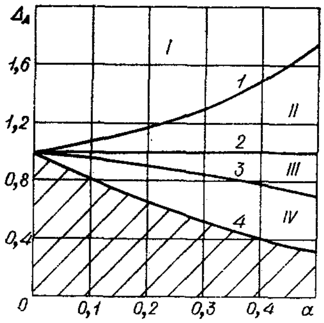

Fig. 2

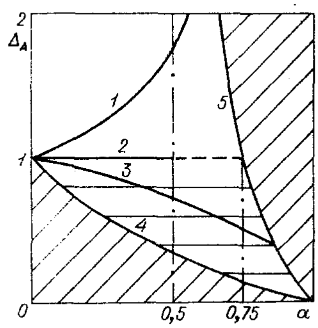

Fig. 3

gularities not characteristic of media belonging to the first group and, in particular, isotropic media [5].

Thus, the two parameters Δ_A and α, by uniquely determining a point of the Δ_A, α plane in Fig. 2, give a complete representation of the anisotropic medium under consideration. Depending on which of the regions I-IV the point occupies, the existence and disposition of the lacunas are determined in the form of the roots of the characteristic equation. The closeness of the boundaries separating the regions determines the size of the lacunas and largely the form of the curves of displacements for the problems just indicated. For example, in the case of copper (Δ_A = 0.478, α = 0.442) the point on the Δ_A, α plane falls into region IV and is situated close to the lower boundary of the region. This indicates that the roots of the characteristic equation belong to the fourth type, the coordinate axes x and z intersect the lacunas, and the lacunas occupy a relatively large region behind the wave front, which is confirmed by specific calculations.

3. The reciprocal of the Young's modulus in the case of cubic crystals is equal to

$$\frac{1}{E} = \frac{c_1 + c_2 - c_3}{(c_1 - c_2 + c_3)(c_1 + 2c_2 - 2c_3)} - \left(\frac{2}{c_1 - c_2 + c_3} - \frac{1}{c_3}\right) L,$$

$$L = \alpha_1^2\alpha_2^2 + \alpha_2^2\alpha_3^2 + \alpha_3^2\alpha_1^2. \tag{3.1}$$

Here α_i are the direction cosines of the sample axis. In terms of Δ_A, α Eq. (3.1) takes the form

$$\frac{1}{E} = \Phi[G + (\Delta_A - 1) L]; \quad \Phi = \frac{(1-\alpha)^2}{\alpha(c_1 - c_3)[\Delta_A(1+\alpha) - (1-\alpha)]};$$

$$G = \frac{\alpha(\Delta_A + 1)\Delta_A}{\Delta_A(1 - 2\alpha) + 2(1-\alpha)}. \tag{3.2}$$

Assuming that $0 < \alpha < 1$, let us consider the Δ_A, α.

On passing through the value $\Delta_{A_{\gamma=0}} = (1-\alpha)/(1+\alpha)$, the function Φ changes sign; this value coincides with the lower limit of permissible values of Δ_A (determined by the hyperbolic conditions), and hence $\Phi > 0$ for the whole permissible range of Δ_A when $0 < \alpha < 1$.

On passing through the value $\Delta_A^{\mp} = -2(1-\alpha)/(1-2\alpha)$, G changes sign; this value divides the region of permissible values of Δ_A into two parts. To the right of the curve $\Delta_A^{\mp}(\alpha)$ G is negative and to the left, positive. The quantity $(\Delta_A - 1)$ is positive for $\Delta_A > 1$ and negative for $\Delta_A < 1$.

Negative values of the Young's modulus correspond to the values of Δ_A lying to the right of the curve $\Delta_A^{\mp}(\alpha)$, and this region should be regarded as forbidden. It is readily seen that in the case of isotropic media (points on the straight line $\Delta_A = 1$) the permissible values include $0 < \alpha < 0.75$ for $\Delta_A < 1$ the Young's modulus assumes maximum values, together with the quantity L, i.e., in the $\langle 111 \rangle$, directions; it assumes minimum values in the direction of the cube axis $\langle 100 \rangle$. For $\Delta_A > 1$ (first group of media) the maximum values of the Young's modulus correspond to the $\langle 100 \rangle$ and the minimum to the $\langle 111 \rangle$ directions.

In Fig. 3 the regions to the right of $\Delta_A = \Delta_A^{\mp}(\alpha)$ and below the curve $\Delta_{A_{\gamma=0}} = (1-\alpha)/(1+\alpha)$, corresponding to the forbidden values of Δ_A are shaded with oblique lines and the region $\Delta_A < 1$ with horizontal lines. Curves 1-4 are the same as in Fig. 2, while curve 5 corresponds to the line $\Delta_A = \Delta_A^{\mp}(\alpha)$. The quantities α, β, γ, the types of lattices (C = cubic, HC = hexagonal close-packed), and the number of the regions (in accordance with Fig. 2) are presented for a number of elements in Table 1.

Table 2 presents data relating to metals for which the value of α lies in the range $0.5 < \alpha < 0.75$.

Let us consider the values of $\alpha > 1$, corresponding to $\Delta_A < 0$. As $\alpha \to \infty$ we have

$$\Delta_{A_{\gamma=0}} = (1-\alpha)/(1+\alpha) \to -1,$$

i.e., for $\alpha > 1$, the hyperbolic conditions allow values Δ_A in the range $-1 < \Delta_A < 0$. Values of $\alpha > 1$ lie to the right of the curve $\Delta_A^{\mp}(\alpha)$ so that for the α values in question we have negative values of the Young's modulus. This also follows from Eqs. (3.2). Thus values of $\alpha > 1$ are forbidden, as are all values of Δ_A, α to the right of curve $\Delta_A = \Delta_A^{\mp}(\alpha)$.

Negative Young's moduli are forbidden for cubic crystals because they fail to satisfy the positive-definite elastic-energy condition for triaxial deformation,

$$a_{11} > 0, \ a_{11} > |a_{12}|, \ a_{11} + 2a_{12} > 0,$$

which in the present notation may be written in the form

$$c_1 > 0, \ c_1 > |c_2 - c_3|, \quad (c_1 + 2c_2 - 2c_3) > 0. \tag{3.3}$$

According to (3.3), in fact, $\Phi G = (c_1 + c_2 - c_3)/[(c_1 - c_2 + c_3)(c_1 + 2c_2 - 2c_3)]$ is always positive and hence so is the Young's modulus.

For media not belonging to the class of cubic crystals, the conditions ensuring a positive Young's modulus do not coincide with the conditions ensuring a positive-definite elastic energy, subject to triaxial deformation. Thus for such media negative values of Young's modulus are no longer forbidden unless they involve infringement of the condition of positive-definite elastic energy.

4. In conclusion, let us indicate the simplest method of finding γ_* which determines the boundary between regions I and II in Figs. 2 and 3.

In addition to the "region-I" medium of current interest, having the wave-front configurations as indicated in Fig. 1b and d, let us consider a model medium differing from the first simply by virtue of the fact that the wave pattern is turned through 45° relative to the coordinate axes. The coordinate axes will then intersect the lacunas, and in accordance with the foregoing classification the model medium will belong to region IV. The transition from region I to region II for the original medium is equivalent to a transition from IV to III for the model. The critical value γ'_* is determined by the equation $\gamma'_*=\alpha'(\alpha'+1)$, where α', γ' are the parameters of the model medium.

The deformation components in the coordinate system rotated through an angle γ are related to those in the original system by the equations

$$\begin{aligned} \varepsilon'_{xx} &= \varepsilon_{xx}\cos^2\lambda + \varepsilon_{zz}\sin^2\lambda + \varepsilon_{xz}\sin\lambda\cos\lambda; \\ \varepsilon'_{zz} &= \varepsilon_{xx}\sin^2\lambda + \varepsilon_{zz}\cos^2\lambda - \varepsilon_{xz}\sin\lambda\cos\lambda; \\ \varepsilon'_{xz} &= -2\varepsilon_{xx}\sin\lambda\cos\lambda + 2\varepsilon_{zz}\sin\lambda\cos\lambda + \varepsilon_{xz}(\cos^2\lambda - \sin^2\lambda). \end{aligned} \tag{4.1}$$

For the angle $\lambda=45°$ $\cos^2\lambda=\sin^2\lambda=1/2$.

For the elastic potential we have two equivalent equations:

$$F = c'_1\left(\varepsilon'^2_{xx} + \varepsilon'^2_{zz}\right) + 2\left(c'_2 - c'_3\right)\varepsilon'_{xx}\varepsilon'_{zz} + c'_3\varepsilon'^2_{xz}; \tag{4.2}$$

$$F = c_1\left(\varepsilon^2_{xx} + \varepsilon^2_{zz}\right) + 2\left(c_2 - c_3\right)\varepsilon_{xx}\varepsilon_{zz} + c_3\varepsilon_{xz}. \tag{4.3}$$

Substituting (4.1) into (4.2) we obtain:

$$F = A\left(\varepsilon^2_{xx} + \varepsilon^2_{zz}\right) + 2B\varepsilon_{xx}\varepsilon_{zz} + C\varepsilon^2_{xz}; \tag{4.4}$$

$$2A = c'_1 + c'_2 + c'_3, \quad 2B = c'_1 + c'_2 - 3c'_3, \quad 2C = c'_1 - c'_2 + c'_3. \tag{4.5}$$

By comparing (4.3) and (4.4) we obtain

$$A=c_1, \; B=c_2 - c_3, \; C=c_3. \tag{4.6}$$

The set of equations (4.5) and (4.6) constitutes a system for determining the coefficients c'_1, c'_2, c'_3 of the model medium; on solving this we have

$$\begin{gathered} c'_1/c_1 = \left[(1+\alpha) + \sqrt{1+\alpha^2-\gamma}\right]/2; \\ c'_3/c_1 = \left[(1+\alpha) - \sqrt{1+\alpha^2-\gamma}\right]/2; \; c'_2/c_1 = 1-\alpha. \end{gathered}$$

For the parameters α', γ' we obtain the following expressions:

$$\alpha' = [(1+\alpha) - \sqrt{1+\alpha^2-\gamma}]/[(1+\alpha)+\sqrt{1+\alpha^2-\gamma}]; \tag{4.7}$$

$$\gamma' = 1 + \alpha'^2 - 4(1-\alpha)^2/\left[(1+\alpha) + \sqrt{1+\alpha^2-\gamma}\right]^2. \tag{4.8}$$

The prime denotes the parameters for the model medium.

Substituting (4.7) and (4.8) into $\gamma'=\alpha'(\alpha'+1)$, we find a quadratic equation for γ:

$$\gamma^2 + 3(1-\alpha)^2 - \alpha(10\alpha^2 - 16\alpha + 10) = 0,$$

and on solving this we obtain the value of $\gamma=\gamma_*$, defined by (1.7). The $\Delta_{A*}(\alpha)$ and $\Delta_A^{\pm}(\alpha)$ curves intersect in a certain point at which $\Delta_A \approx 2.6$. Thus in all cases when $\Delta_A > 2.6$ the geometry of the wave fronts takes the form of Fig. 1b,c,d, i.e., u is characterized by the presence of four lacunas not lying on the coordinate axes.

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